IX. *On the General Resolution of Algebraical Equations.*
by Edward Waring, *M. D. F. R. S. and of the Institute*
of Bononia, *Lucasian Professor of Mathematics in the*
University of Cambridge.

IN the year 1757 I sent some papers to the Royal So-
ciety, which papers were printed in the year 1759,
and copies of them delivered to several persons; these
papers somewhat corrected, with the addition of a se-
cond part on the properties of curve lines, were pub-
lished in the year 1762. In the years 1767, 1768
and 1769 I printed, and published in the beginning of
the year 1770, the same papers with additions and e-
mendations under the title of Meditationes Algebraicæ.
In these papers were contained, with many other inven-
tions, the most general resolution of algebraical equa-
tions known, as it contains the resolution of every al-
gebraical equation, of which the general resolution has
been given, viz. the resolution of quadratic, cubic and
biquadratic, the resolution of Mr. DE MOIVRE's and Mr.
BEROUT's (since published) equations; it discovers the

refolution

resolution of an equation of n dimensions, of which the n roots are given, and also deduces innumerable e-quations of n dimensions, which contain $n - 1$ independent coefficients. From whence it seems probable, that this new method of mine may contain the most general resolution of algebraical equations that ever has, or, perhaps, ever will be invented.

The general resolution is $x = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + d\sqrt[n]{p^4}.... + r\sqrt[n]{p^{n-3}} + s\sqrt[n]{p^{n-2}} + t\sqrt[n]{p^{n-1}} + \frac{A}{n}$, if the equation be $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} -$ &c. $= 0$.

I shall add the resolution of some particular equations from this method, and then subjoin the equation to which $x = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} +$ &c. is the general resolution.

1. Let the resolution be $x = a\sqrt[3]{p} + b\sqrt[3]{p^2}$, and the cor-respondent equation free from radicals will be found $x^3 - 3abpx - a^3p - b^3p^2 = 0$. Let $x^3 - Px - Q = 0$ be a cubic equation whose resolution is required, which suppose the same as the equation found above, and consequently their correspondent terms equal, i. e. $P = 3abp$ and $Q = a^3p + b^3p^2$, whence $p = \frac{P}{3ab}$, which value being substituted for p in the second equation, there results $Q = \frac{Pa^2}{3b} + \frac{bP^2}{9a^2}$. In this equation for a or b may be assumed unity, or any other

other quantity whatever, and there will refult an equa-
tion of the formula of a quadratic from which the other
b or a may be found, whence from the equation ($p =$
$\frac{P}{3ab}$) p may be deduced, and confequently the refolution
of the cubic required.

In the fame manner for p may be affumed any quan-
tity whatever, and in the equation $Q = a^3 p + b^3 p^2$ for b
fubftitute its value $\frac{P}{3ap}$, or for a its value $\frac{P}{3bp}$, and there
refult the equations $Q = a^3 p + \frac{P^3}{27a^3 p^2}$, and $Q = \frac{P^3}{27b^3 p^2} + b^3 p^2$,
which have the formula of a quadratic, from which
may be deduced the refolution of the cubic required.

2. Let the refolution affumed be $x = a\sqrt[4]{p} + b\sqrt[4]{p^2} +$
$c\sqrt[4]{p^3}$; exterminate the irrational quantities, and there
refults the equation $x^4 - (2b^2 + 4ac)px^2 - 4(a^2 bp + bc^2 p^2)x -$
$a^4 p + b^4 p^2 - c^4 p^3 + 2a^2 c^2 p^2 - 4ab^2 cp^2 = 0$; fuppofe $p = 1$, and
the given equation $x^4 + qx^2 - rx + s = 0$, let the correfpon-
dent terms of the given and refulting equations be re-
fpectively made equal to each other, and there refult
the three equations $2b^2 + 4ac = -q$, $4b(a^2 + c^2) = r$, and
$a^4 - b^4 + c^4 - 2a^2 c^2 + 4ab^2 c = -s$; reduce thefe equations in-
to one, fo that the unknown quantities a and c may be
exterminated, and there refults the equation $4b^6 + 2qb^4$
$+ (\frac{q^2}{4} - s) b^2 - \frac{r^2}{16} = 0$ of the formula of a cubic, from which

the

the unknown quantity b may be found, which being substituted for its value (b) in the preceding equations, from the equations thence ensuing may be found the unknown quantities a and c, and consequently the resolution of the given biquadratic $x^4 + qx^2 - rx + s = 0$.

From the same principles can be deduced different resolutions of the above-mentioned biquadratic $x^4 + qx^2 - rx + s = 0$.

3. I. Let $x = a\sqrt[2n]{p} + b\sqrt[2n]{p^2}$, then will the equation free from radicals be $x^{2n} - 2b^n p x^n - \dfrac{n}{1.2} \cdot 2nb^{n-1}a^2 p x^{n-1} - $

$\dfrac{n \times \overline{n^2-1}}{1.2.3.4} \times 2nb^{n-2}a^4 p x^{n-2} - \dfrac{n \times \overline{n^2-1} \times \overline{n^2-4}}{1.2.3.4.5.6} \times 2nb^{n-3}a^6 p x^{n-3} - $

$\dfrac{n.\overline{n^2-1}.\overline{n^2-4}.\overline{n^2-9}}{1.2.3.4.5.6.7.8} \times 2nb^{n-4}a^8 p x^{n-4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad - $

$\dfrac{n.\overline{n^2-1} \times \overline{n^2-4} \times \overline{n^2-9} . \overline{n^2-16}...\overline{n^2-\overline{n-2}^2}}{1.2.3.4.5.6.7....2n-2} \times 2na^{2n-2}bpx = a^n p - b^n p^2.$

This equation may be deduced from the following principles. Let α, β, γ, δ, ε, &c. be the $2n$ roots of the equation $z^{2n} - 1 = 0$, then (by Prop. XXIII. of my Meditat. Algebraicæ) the equation free from radicals will be the product of the following quantities $(x - a\alpha\sqrt[2n]{p} - b\alpha^2\sqrt[2n]{p^2})$ $(x - a\beta\sqrt[2n]{p} - b\beta^2\sqrt[2n]{p^2})$ $(x - a\gamma\sqrt[2n]{p} - b\gamma^2\sqrt[2n]{p^2})$ $(x - a\delta\sqrt[2n]{p} - b\delta^2\sqrt[2n]{p^2})$ $(x - a\varepsilon\sqrt[2n]{p} - b\varepsilon^2\sqrt[2n]{p^2})$ &c. $= 0$: multiply these quantities into each other, and from the resulting product, by Prob. III. of the Meditat. Algebr. easily can

      be

be deduced the equation free from radicals which was to be found.

3. II. Let $x = a\sqrt[2n+1]{p} + b\sqrt[2n+1]{p^2}$, then will the correspondent equation free from radicals be $x^{2n+1} - \overline{2n+1}\, b^n a p x^n - \frac{n \cdot n+1}{1 \cdot 2 \cdot 3}$

$\times \overline{2n+1}\, b^{n-1} a^3 p x^{n-1} - \frac{n \times \overline{n^2-1} \times \overline{n+2}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \times \overline{2n+1}\, b^{n-2} a^5 p x^{n-2} -$

$\frac{n \times \overline{n^2-1} \times \overline{n^2-4} \times \overline{n+3}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \times \overline{2n+1}\, b^{n-3} a^7 p x^{n-3} - \frac{n \cdot n^2 - 1 \cdot n^2 - 4 \cdot n^2 - 9 \cdot n + 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} \times$

$\overline{2n+1}\, b^{n-4} a^9 p x^{n-4} - \ldots - \frac{n \cdot n^2 - 1 \cdot n^2 - 4 \cdot n^2 - 9 \cdot n^2 - 16 \ldots n^2 - n - 2^2 \times \overline{2n-1}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \ldots 2n - 1} \times$

$\overline{2n+1}\, b a^{2n-1} p x = a^{2n+1} p + b^{2n+1} p^2.$

This may be derived from the same principles as the preceding.

3. III. In general let the equation be $x = a\sqrt[m]{p} + b\sqrt[m]{p^2}$, then will the equation free from radicals become $x^m - m a^{m-2} b p x - m \cdot \frac{m-3}{2} a^{m-4} b^2 p x^2 - m \cdot \frac{m-4}{2} \cdot \frac{m-5}{3} a^{m-6} b^3 p x^3 - m \cdot \frac{m-5}{2} \cdot \frac{m-6}{3} \cdot \frac{m-7}{4} a^{m-8} b^4 p x^4 - m \cdot \frac{m-6}{2} \cdot \frac{m-7}{3} \cdot \frac{m-8}{4} \cdot \frac{m-9}{5} a^{m-10} b^5 p x^6 - \&c. = a^m p \pm b^m p^2$; if m denotes an even number, it will be $- b^m p^2$, but if an odd number, it will be $+ b^m p^2$.

4. I. Let n denote an odd number, and $x = a\sqrt[n]{p} + b\sqrt[n]{p^3}$, then will $x^n - p\, (n a^{n-3} b x^2 + n \cdot \frac{n-5}{2} a^{n-6} b^2 x^4 + n \cdot \frac{n-7}{2} \cdot \frac{n-8}{3} a^{n-9} b^3 x^5 + n \cdot \frac{n-9}{2} \cdot \frac{n-10}{3} \cdot \frac{n-11}{4} a^{n-12} b^4 x^8 + n \cdot \frac{n-11}{2} \cdot \frac{n-12}{3} \cdot \frac{n-13}{4} \cdot \frac{n-14}{5} a^{n-15} b^5 x^{10} + n \cdot \frac{n-13}{2} \cdot \frac{n-14}{3} \cdot \frac{n-15}{4} \cdot \frac{n-16}{5} \cdot \frac{n-17}{6} a^{n-18} b^6 x^{12} + n \cdot$

$$\frac{n-15}{2}\cdot\frac{n-16}{3}\cdot\frac{n-17}{4}\cdot\frac{n-18}{5}\cdot\frac{n-19}{6}\cdot\frac{n-20}{7}\; a^{n-21}b^7x^{14} + \&c.) \pm p^2\,(na^{\frac{n-3}{2}}$$

$$b^{\frac{n+1}{2}}x - \frac{1}{2^2}\times n.\frac{n-5}{2}\cdot\frac{n-7}{3}\; a^{\frac{n-9}{2}}b^{\frac{n+3}{2}}x^3 + \frac{1}{2^4}\times n.\frac{n-7}{2}\cdot\frac{n-9}{3}\cdot\frac{n-11}{4}\cdot\frac{n-13}{5}$$

$$a^{\frac{n-15}{2}}b^{\frac{n+5}{2}}x^5 - \frac{1}{2^6}\times n.\frac{n-9}{2}\cdot\frac{n-11}{3}\cdot\frac{n-13}{4}\cdot\frac{n-15}{5}\cdot\frac{n-17}{6}\cdot\frac{n-19}{7}\; a^{\frac{n-21}{2}}b^{\frac{n+7}{2}}x^7 +$$

$$\frac{1}{2^8}\times n.\frac{n-11}{2}\cdot\frac{n-13}{3}\cdot\frac{n-15}{4}\cdot\frac{n-17}{5}\cdot\frac{n-19}{6}\cdot\frac{n-21}{7}\cdot\frac{n-23}{8}\cdot\frac{n-25}{9}\; a^{\frac{n-27}{2}}b^{\frac{n+9}{2}}x^9 -$$

$$\&c.) = a^np + b^np^3.$$

The quantity $\pm p^2$ denotes $+p^2$ if $\frac{n-3}{4}$ is a whole number, otherwise $-p^2$.

4. II. Let n denote an even number, and x as before $= a\sqrt[n]{p} + b\sqrt[n]{p^3}$, then will $x^n - p\,(na^{n-3}bx^2 + n.\frac{n-5}{2}\,a^{n-6}b^2x^4$

$+ n.\frac{n-7}{2}\cdot\frac{n-8}{3}\,a^{n-9}b^3x^6 + n.\frac{n-9}{2}\cdot\frac{n-10}{3}\cdot\frac{n-11}{4}\,a^{n-12}b^4x^8 + n.\frac{n-11}{2}.$

$\frac{n-12}{3}\cdot\frac{n-13}{4}\cdot\frac{n-14}{5}\,a^{n-15}b^5x^{10} + \&c.) \pm p^2\left(\frac{1}{2}n.\frac{n-4}{2}\,a^{\frac{n-6}{2}}b^{\frac{n+2}{2}}x^2 - \right.$

$\frac{1}{2^3}\times n.\frac{n-6}{2}\cdot\frac{n-8}{3}\cdot\frac{n-10}{4}\,a^{\frac{n-12}{2}}b^{\frac{n+4}{2}}x^4 + \frac{1}{2^5}\times n.\frac{n-8}{2}\cdot\frac{n-10}{3}\cdot\frac{n-12}{4}\cdot\frac{n-14}{5}.$

$\frac{n-16}{6}\,a^{\frac{n-18}{2}}b^{\frac{n+6}{2}}x^6 - \frac{1}{2^7}\times n.\frac{n-10}{2}\cdot\frac{n-12}{3}\cdot\frac{n-14}{4}\cdot\frac{n-16}{5}\cdot\frac{n-18}{6}\cdot\frac{n-20}{7}\cdot\frac{n-22}{8}$

$a^{\frac{n-24}{2}}b^{\frac{n+8}{2}}x^8 + \&c.) = a^np + b^np^3 \pm 2a^{\frac{n}{2}}b^{\frac{n}{2}}p^2.$

The quantities $\pm p^2$ and $\pm 2a^{\frac{n}{2}}b^{\frac{n}{2}}p^2$ denote $+p^2$, and $+2a^{\frac{n}{2}}b^{\frac{n}{2}}p^2$, if $\frac{n-2}{4}$ is a whole number, otherwise they denote $-p^2$ and $-2a^{\frac{n}{2}}b^{\frac{n}{2}}p^2$ respectively.

5. I. Let $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-1}}$, and n an odd number,

then

then will $x^n - nabpx^{n-2} + n.\frac{n-3}{2}a^2b^2p^2x^{n-4} - n.\frac{n-4}{2}.\frac{n-5}{3}.$
$a^3b^3p^3x^{n-6} + n.\frac{n-5}{2}\frac{n-6}{3}.\frac{n-7}{4}a^4b^4p^4x^{n-8} - \&c. = a^np + b^np^{n-1}.$

5. II. Let $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-1}}$, and n an even number, then will $x^n - nabpx^{n-2} + n.\frac{n-3}{2}a^2b^2p^2x^{n-4} - n.\frac{n-4}{2}.$
$\frac{n-5}{3}a^3b^3p^3x^{n-6} + n.\frac{n-5}{2}.\frac{n-6}{3}.\frac{n-7}{4}a^4b^4p^4x^{n-8} - \&c. = a^np \mp$
$2a^{\frac{n}{2}}b^{\frac{n}{2}}p^{\frac{n}{2}} + b^np^{n-1}$; it will be $+ 2a^{\frac{n}{2}}b^{\frac{n}{2}}p^{\frac{n}{2}}$ if $n = 4r + 2$;
but $- 2a^{\frac{n}{2}}b^{\frac{n}{2}}p^{\frac{n}{2}}$ if $n = 4r$.

6. I. Let $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-2}}$, and n an odd number, which has not the number 3 for a divisor, then will
$x^n - na^2bpx^{n-3} + n.\frac{n-5}{2}a^4b^2p^2x^{n-6} - n.\frac{n-7}{2}.\frac{n-8}{3}a^6b^3p^3x^{n-9} +$
$n.\frac{n-9}{2}.\frac{n-10}{3}.\frac{n-11}{4}a^8b^4p^4x^{n-12} - n.\frac{n-11}{2}.\frac{n-12}{3}.\frac{n-13}{4}.\frac{n-14}{5}$
$a^{10}b^5p^5x^{n-15} + n.\frac{n-13}{2}.\frac{n\times14}{3}.\frac{n-15}{4}.\frac{n-16}{5}.\frac{n-17}{6}a^{12}b^6p^6x^{n-6} - \&c.$
(to m terms, where m is the number either equal to, or the least greater than $\frac{n}{3}$) $- ab^{\frac{n+1}{2}}p^{\frac{n-1}{2}}(nx^{\frac{n-3}{2}} + \frac{1}{4}n.\frac{n-5}{2}.\frac{n-7}{3}$
$a^2bpx^{\frac{n-9}{2}} + \frac{1}{2^4}n.\frac{n-7}{2}.\frac{n-9}{3}.\frac{n-11}{4}.\frac{n-13}{5}a^4b^2p^2x^{\frac{n-15}{2}} + \frac{1}{2^6}\times n.\frac{n-9}{2}.$
$\frac{n-11}{3}.\frac{n-13}{4}.\frac{n-15}{5}.\frac{n-17}{6}.\frac{n-19}{7}a^6b^3p^3x^{\frac{n-21}{2}} + \frac{1}{2^8}\times n.\frac{n-11}{2}.\frac{n-13}{3}.\frac{n-15}{4}.$
$\frac{n-17}{5}.\frac{n-19}{6}.\frac{n-21}{7}.\frac{n-23}{8}.\frac{n-25}{9}a^8b^4p^4x^{\frac{n-27}{2}} + \&c.) = A = a^np +$
$b^np^{n-2}.$

Let n be an odd number divisible by 3, then will the
above-

above-mentioned quantity $= \mathrm{A} = a^n p + b^n p^{n-2} + 3a^{\frac{n}{3}} b^{\frac{2n}{3}} p^{\frac{2n}{3}-1} + 3a^{\frac{2n}{3}} b^{\frac{n}{3}} p^{\frac{n}{3}}.$

6. II. Let n be an even number, not divisible by 3, then will $x^n - na^2 bp x^{n-3} + n.\frac{n-5}{2} a^4 b^2 p^2 x^{n-6} - n.\frac{n-7}{2}.\frac{n-8}{3}$ $a^6 b^3 p^3 x^{n-9} + n.\frac{n-9}{2}.\frac{n-10}{3}.\frac{n-11}{4} a^8 b^4 p^4 x^{n-12} - n.\frac{n-11}{2}.\frac{n-12}{3}.\frac{n-13}{4}.$ $\frac{n-14}{5} a^{10} b^5 p^5 x^{n-15} + n.\frac{n-13}{2}.\frac{n-14}{3}.\frac{n-15}{4}.\frac{n-16}{5}.\frac{n-17}{6} a^{12} b^6 p^6 x^{n-18} -$ &c. to m terms as before $- b^{\frac{n}{2}} p^{\frac{n}{2}-1} (2x^{\frac{n}{2}} + \frac{1}{2} n.\frac{n-4}{2}$ $a^2 bp x^{\frac{n}{2}-3} + \frac{1}{2^3} \times n.\frac{n-6}{2}.\frac{n-8}{3}.\frac{n-10}{4} a^4 b^2 p^2 x^{\frac{n}{2}-6} + \frac{1}{2^5} \times n.\frac{n-8}{2}.\frac{n-10}{3}.$ $\frac{n-12}{4}.\frac{n-14}{5}.\frac{n-16}{6} a^6 b^3 p^3 x^{\frac{n}{2}-9} + \frac{1}{2^7} \times n.\frac{n-10}{2}.\frac{n-12}{3}.\frac{n-14}{4}.\frac{n-16}{5}.\frac{n-18}{6}.$ $\frac{n-20}{7}.\frac{n-22}{8} a^8 b^4 p^4 x^{\frac{n}{2}-12} + \text{&c.}) = \mathrm{A} = a^n p - b^n p^{n-2}.$

Let n be an even number divisible by 3, then will the above-mentioned quantity $\mathrm{A} = a^n p - b^n p^{n-2} - 3a^{\frac{2n}{3}} b^{\frac{n}{3}} p^{\frac{n}{3}} + 3a^{\frac{n}{3}} b^{\frac{2n}{3}} p^{\frac{2n}{3}-1}.$

In all the preceding cases n, m and r denote whole affirmative numbers.

These equations may be deduced in the same manner as is before given in Case 3. I; or can be demonstrated by writing in the equation free from radicals for the different powers of x their values deduced from the given equation $x = a\sqrt[m]{p} + b\sqrt[m]{p^n}.$

To render the folution general, it may not be impro-
per to fubjoin the fubfequent.

L E M M A.

1. Let $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$, &c. be the refpective roots of the
equation $z^n - 1 = 0$; then will $\alpha^m + \beta^m + \gamma^m + \delta^m + \varepsilon^m +$&c.$= 0$,
unlefs $n = m$, or n is a divifor of m, in which cafe $\alpha^m +$
$\beta^m + \gamma^m + \delta^m + \varepsilon^m +$ &c. $= n$.

2. The fum of all quantities of the following kind
$\alpha^m\beta^r + \alpha^r\beta^m + \alpha^m\gamma^r + \alpha^r\gamma^m + \beta^m\gamma^r + \beta^r\gamma^m + \alpha^m\delta^r +$ &c. will be
$= 0$; unlefs n be either equal to, or a divifor of $m + r$,
in which cafe the fum above-mentioned will be $= -n$;
except n be either equal to m or r, or a divifor of them,
in which cafe the fum will be $n^2 - n$; but if $m = r$, then
in the former cafe will the above-mentioned fum $= -\frac{n}{2}$,
and in the latter $= \frac{n^2 - n}{2}$.

3. The fum of all quantities of this kind $\alpha^m\beta^r\gamma^s\delta^t$ &c. $+$
$\alpha^r\beta^m\gamma^s\delta^t$ &c. $+ \alpha^m\beta^r\gamma^s\delta^t$ &c. $+ \alpha^r\beta^s\gamma^m\delta^t$ &c. $+$ &c. will be
$= 0$, unlefs n be either equal to $r + m + s + t$ &c. or a di-
vifor of it.

Let π be the number of indices m, r, s, t, &c. and
n be either equal to $m + r + s + t +$ &c. or a divifor of it,
but n be neither equal to, nor a divifor of the fum of
any two, three, four, ... $\pi - 3$, $\pi - 2$ or $\pi - 1$ of the above-
mentioned

mentioned quantities; then will the fum above-mentioned $= \mp \overline{1.2 \ 3.4 \ldots \pi - 2} \cdot \overline{\pi - 1} \times n$; where it will be $+$, if π be an odd number; otherwife $-$.

In this cafe, if a indices be m, b indices be r, c indices be s, d indices be t, &c. then will the above-mentioned fum $= \mp \dfrac{1.2 \ 3.4 \ldots \overline{\pi - 2} \times \overline{\pi - 1}}{1.2.3 \ldots a \times 1.2.3 \ldots b \times 1.2.3 \ldots c \times 1.2.3 \ldots d \times \&c.} \times n.$

Let n be either equal to, or a divifor of the fum of any number ϱ (lefs than π) of the above-mentioned quantities m, r, s, t, &c. and confequently either equal to, or a divifor of the fum of the $(\pi - \varrho)$ remaining quantities: find the fum of all poffible quantities of this kind $\overline{1.2.3 \ldots \varrho - 2} \times \overline{\varrho - 1} \times \overline{1.2.3 \ldots \pi - \varrho - 2} \times \overline{\pi - \varrho - 1} \times n^2$, which fum call A.

Let n be either equal to, or a divifor of the fum of any number (σ) of the above-mentioned quantities m, r, s, t, &c; and alfo equal to, or a divifor of the fum of any number (ϱ') of the remaining quantities, and confequently it will be either equal to, or a divifor of, the fum of the $(\pi - \varrho' - \sigma)$ remaining quantities; then find the fum of all poffible quantities of this fort $\overline{1.2.3 \ldots \sigma - 2} \times \overline{\sigma - 1} \times \overline{1.2.3 \ldots \varrho' - 2} \times \overline{\varrho' - 1} \times \overline{1.2.3 \ldots \pi - \varrho' - \sigma - 2}$ $\times \overline{\pi - \varrho' - \sigma - 1} \times n^3$, which fum call B.

In the fame manner let n be either equal to, or a divifor of the fum of any number (τ) of the above-mentioned

tioned quantities m, r, s, t, &c; and fimilarly let n be either equal to, or a divifor of the fum of any number (σ') of the remaining quantities; and alfo let n be either equal to, or a divifor of the fum of any number (ϱ'') of the remaining quantities; then will n be either equal to, or a divifor of the fum of the $(\pi-\tau-\sigma'-\varrho'')$ remaining quantities: find the fum of all quantities of this fort

$$1.2.3..\overline{\tau-2} \times \overline{\tau-1} \times 1.2.3..\overline{\sigma'-2} \times \overline{\sigma'-1} \times 1.2.3..\overline{\varrho''-2} \times$$
$$\overline{\varrho''-1} \times 1.2.3..\overline{\pi-\tau-\sigma'-\varrho''-2} \times \overline{\pi-\tau-\sigma'-\varrho''-1} \times n^4, \text{ which}$$

fum call c; and fo on; then will the above-mentioned fum $\alpha^m\beta^r\gamma^s\delta^t$ &c. $+ \alpha^r\beta^m\gamma^s\delta^t$ &c. $+ \alpha^m\beta^s\gamma^r\delta^s$ &c. $+ \alpha^r\beta^s\gamma^m\delta^t$ &c. $+$ &c. $= \mp (1.2.3..\overline{\pi-2} \times \overline{\pi-1} \times n - A + B - C + D - \text{&c.})$ where it will be $+$ if π be an odd number, otherwife $-$.

In this cafe, if a indices be m, b indices be r, c indices be s, d indices be t, &c. then will the above-mentioned fum $= \mp \frac{1.2.3..\overline{\pi-2} \times \overline{\pi-1}.n - A + B - C + D - \text{&c.}}{1.2.3..a \times 1.2.3..b \times 1.2.3..c \times 1.2.3..d \times \text{&c.}}$.

7. Let α, β, γ, δ, ε, &c. be the roots of the equation $z^n - 1 = 0$, and the refolution be $x = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + d\sqrt[n]{p^4}..... + b\sqrt[n]{p^\lambda}... + k\sqrt[n]{p^\mu}.... + l\sqrt[n]{p^\nu}... + q\sqrt[n]{p^\xi}.... + r\sqrt[n]{p^\phi}.... + s\sqrt[n]{p^{n-4}} + t\sqrt[n]{p^{n-3}} + v\sqrt[n]{p^{n-2}} + u\sqrt[n]{p^{n-1}}$; then will the different values of x be refpectively $a\sqrt[n]{p} \times \alpha + b\sqrt[n]{p^2} \times \alpha^2 + c\sqrt[n]{p^3} \times \alpha^3 + d\sqrt[n]{p^4} \times \alpha^4.... + b\sqrt[n]{p^\lambda} \times \alpha^\lambda... + k\sqrt[n]{p^\mu} \times \alpha^\mu...... + s\sqrt[n]{p^{n-4}} \times \alpha^{n-4} + t\sqrt[n]{p^{n-3}} \times \alpha^{n-3} + v\sqrt[n]{p^{n-2}} \times \alpha^{n-2} + u\sqrt[n]{p^{n-1}} \times \alpha^{n-1}$;

$$a\sqrt[n]{p} \times \beta + b\sqrt[n]{p^2} \times \beta^2 + c\sqrt[n]{p^3} \times \beta^3 + d\sqrt[n]{p^4} \times \beta^4 \ldots + b\sqrt[n]{p^\lambda} \times$$
$$\beta^\lambda \ldots + k\sqrt[n]{p^\mu} \times \beta^\mu \ldots\ldots + s\sqrt[n]{p^{n-4}} \times \beta^{n-4} + t\sqrt[n]{p^{n-3}} \times \beta^{n-3} +$$
$$v\sqrt[n]{p^{n-2}} \times \beta^{n-2} + u\sqrt[n]{p^{n-1}} \times \beta^{n-1} \; ;$$

$$a\sqrt[n]{p} \times \gamma + b\sqrt[n]{p^2} \times \gamma^2 + c\sqrt[n]{p^3} \times \gamma^3 + d\sqrt[n]{p^4} \times \gamma^4 \ldots + b\sqrt[n]{p^\lambda} \times$$
$$\gamma^\lambda \ldots + k\sqrt[n]{p^\mu} \times \gamma^\mu \ldots\ldots + s\sqrt[n]{p^{n-4}} \times \gamma^{n-4} + t\sqrt[n]{p^{n-3}} \times \gamma^{n-3} +$$
$$v\sqrt[n]{p^{n-2}} \times \gamma^{n-2} + u\sqrt[n]{p^{n-1}} \times \gamma^{n-1} \; ;$$

$$a\sqrt[n]{p} \times \delta + b\sqrt[n]{p^2} \times \delta^2 + c\sqrt[n]{p^3} \times \delta^3 + d\sqrt[n]{p^4} \times \delta^4 \ldots + b\sqrt[n]{p^\lambda} \times \delta^\lambda \ldots$$
$$+ k\sqrt[n]{p^\mu} \times \delta^\mu \ldots\ldots + s\sqrt[n]{p^{n-4}} \times \delta^{n-4} + t\sqrt[n]{p^{n-3}} \times \delta^{n-3} + v\sqrt[n]{p^{n-2}} \times$$
$$\delta^{n-2} \div u\sqrt[n]{p^{n-1}} \times \delta^{n-1} \; ;$$

&c. &c. &c. &c.

and consequently the sum of the values or roots, which is the coefficient of the second term of the equation sought, will be $a\sqrt[n]{p} \times \overline{\alpha + \beta + \gamma + \delta + \&c.}$ $(o) + b\sqrt[n]{p^2} \times \overline{\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \&c.}$ $(o) + c\sqrt[n]{p^3} \times \overline{\alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \&c.}$ $(o) + \ldots + v\sqrt[n]{p^{n-2}} \times \overline{\alpha^{n-2} + \beta^{n-2} + \gamma^{n-2} + \delta^{n-2} + \&c.}$ $(o) + u\sqrt[n]{p^{n-1}} \times \overline{\alpha^{n-1} + \beta^{n-1} + \gamma^{n-1} + \delta^{n-1} + \&c.}$ $(o) = 0$.

The sum of the products of every two of the values or roots, which is the coefficient of the third term of the equation sought, will be $a^2\sqrt[n]{p^2} \times \overline{\alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta +}$ $\overline{\beta\delta + \gamma\delta + \&c.}$ $(o) + ab\sqrt[n]{p^3} \times \overline{\alpha\beta^2 + \beta\alpha^2 + \alpha\gamma^2 + \gamma\alpha^2 + \beta\gamma^2 + \gamma\beta^2 +}$ $\overline{\alpha\delta^2 + \delta\alpha^2 + \&c.}$ (o), and in general all the terms will be 0, unless $a \times u \times p \times \overline{\alpha\beta^{n-1} + \beta\alpha^{n-1} + \alpha\gamma^{n-1} + \gamma\alpha^{n-1} + \beta\gamma^{n-1} +}$

$$\overline{\gamma\beta^{n-1}+\alpha\delta^{n-1}+\delta\alpha^{n-1}+\beta\delta^{n-1}+\delta\beta^{n-1}+\gamma\delta^{n-1}+\delta\gamma^{n-1}+\&c.}(-n)$$

$$+b\times v\times p\times\overline{\alpha^2\beta^{n-2}+\beta^2\alpha^{n-2}+\alpha^2\gamma^{n-2}+\gamma^2\alpha^{n-2}+\beta^2\gamma^{n-1}+\gamma^2\beta^{n-2}}$$

$$\overline{+\alpha^2\delta^{n-2}+\delta^2\alpha^{n-2}+\beta^2\delta^{n-2}+\delta^2\beta^{n-2}+\&c.}\ (-n)+ctp\times$$

$$\overline{\alpha^3\beta^{n-3}+\beta^3\alpha^{n-3}+\alpha^3\gamma^{n-3}+\gamma^3\alpha^{n-3}+\beta^3\gamma^{n-3}+\gamma^3\beta^{n-3}+\alpha^3\beta^{n-3}+}$$

$$\overline{\delta^3\alpha^{n-3}+\beta^3\delta^{n-3}+\delta^3\beta^{n-3}+\&c.}\ (-n)+dsp\times\overline{\alpha^4\beta^{n-4}+\beta^4\alpha^{n-4}+}$$

$$\overline{\alpha^4\gamma^{n-4}+\gamma^4\alpha^{n-4}\beta^4\gamma^{n-4}+\gamma^4\beta^{n-4}+\&c.}\ (-n)+\&c.=-np(au+$$

$$bv+ct+ds+\&c.)$$

If $n=2\lambda$, then will the coefficient of b^2p be $\frac{n}{2}$, i.e.
the above-mentioned coefficient will be $-np\,(au+bv+ct+ds+....+\frac{1}{2}b^2)$.

The fum of the contents of every three of the a-
bove-mentioned values or roots, which is the coefficient
of the fourth term of the equation required, will be·

$$a^3\sqrt[n]{p^3}\times\overline{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta+\&c.}\ (o)+a^2b\sqrt[n]{p^4}\times$$

$$\overline{\alpha\beta\gamma^2+\alpha\gamma\beta^2+\beta\gamma\alpha^2+\alpha\beta\delta^2+\&c.}\ (o)+\&c.+a^2v\sqrt[n]{p^n}\times$$

$$\overline{\alpha\beta\gamma^{n-2}+\alpha\gamma\beta^{n-2}+\beta\gamma\alpha^{n-2}+\alpha\beta\delta^{n-2}+\alpha\delta\beta^{n-2}+\&c.}\ (\tfrac{1.2.n}{1.2})+$$

$$abt\sqrt[n]{p^n}\times\overline{\alpha\beta^2\gamma^{n-3}+\alpha\gamma^2\beta^{n-3}+\beta\alpha^2\gamma^{n-3}+\beta\gamma^2\alpha^{n-3}+\gamma\alpha^2\beta^{n-3}+}$$

$$\overline{\gamma\beta^2\alpha^{n-3}+\alpha\beta^2\delta^{n-3}}\ \&c.\ (1.2.n)+\&c;.$$ and in general all
the terms (unlefs the quantity $\sqrt[n]{p^\theta}$ contained in the
term have this formula $\sqrt[n]{p^n}=p$, or $\sqrt[n]{p^{2n}}=p^2$) will be$=o$;
let the general term be denoted by $blk\sqrt[n]{p^{\lambda+\mu+\nu}}\times$

$$\overline{\alpha^\lambda\beta^\mu\gamma^\nu+\alpha^\lambda\beta^\nu\gamma^\mu+\alpha^\mu\beta^\lambda\gamma^\nu+\alpha^\mu\beta^\nu\gamma^\lambda+\alpha^\nu\beta^\mu\gamma^\lambda+\alpha^\nu\beta^\lambda\gamma^\mu+\alpha^\lambda\beta^\mu\delta^\nu+\&c.}$$

firft let $\lambda+\mu+\nu$ neither be equal to n or $2n$, then will the
term

term above-mentioned $= 0$; if it be equal to n or $2n$, then will the term be $1.2 \times n \times bklp$ or $1.2nbklp^2$.

If two of the three indexes λ, μ, ν be equal to each other, then divide the above-mentioned term by 1.2; if the three indexes be equal, i. e. $\lambda = \mu = \nu$, divide it by $1.2.3$: find all quantities of this kind where $\lambda + \mu + \nu$ either is equal to n or $2n$, and add all the term from thence derived, and call the fum of them A.

The fum of the contents of every four of the values or roots above-mentioned, which is the coefficient of the fourth term of the equation required, will be

$$a^4 \cdot \sqrt[n]{p^4} \times \overline{\alpha\beta\gamma\delta + \alpha\beta\gamma\delta} + \&c. \ (0) \ + a^3 b \cdot \sqrt[n]{p^5} \times \overline{\alpha\beta\gamma\delta^2 + \alpha\beta\gamma^2\delta +}$$
$$\overline{\alpha\beta^2\gamma\delta + \alpha^2\beta\gamma\delta} + \&c. \ (0) \ + \&c: \ \text{let} \ bklq \sqrt[n]{p^{\lambda+\mu+\nu+\xi}} \times$$
$$\overline{\alpha^\lambda\beta^\mu\gamma^\nu\delta^\xi + \alpha^\lambda\beta^\mu\gamma^\xi\delta^\nu + \alpha^\lambda\beta^\nu\gamma^\mu\delta^\xi + \alpha^\lambda\beta^\nu\gamma^\xi\delta^\mu + \alpha^\lambda\beta^\xi\gamma^\nu\delta^\mu + \alpha^\lambda\beta^\xi\gamma^\mu\delta^\nu +}$$
$$\overline{\alpha^\mu\beta^\lambda\gamma^\nu\delta^\xi} + \&c. \ \text{denote a general term; this term will be}$$

$= 0$, unlefs $\lambda + \mu + \nu + \xi$ either $= n$ or $2n$ or $3n$; in which cafe the term will be either $-1.2.3nbklqp$ or $-1.2.3nbklqp^2$ or $-1.2.3nbklqp^3$; unlefs $\lambda + \mu = \nu + \xi = n$, when the above-mentioned term will be $-(1.2.3n - n^2) bklqp^2$; in this cafe if $\lambda = \nu$, and confequently $\mu = \xi$, then it will be $-(1.2.3n - 1.2n^2) bklqp^2$; but if $\lambda = \mu = \nu = \xi = \frac{n}{2}$, then will the term be $-(1.2.3n - 3n^2) bklqp^2$.

In all thefe cafes, if two of the indexes λ, μ, ν, ξ be equal, then muft the term given above be divided by

1.2; if three, by 1.2.3; if four, by 1.2.3.4; and laſtly if two are equal to each other, and the two remaining indexes equal to each other, but not to the former two, then muſt the term aforeſaid be divided by 1.2.1.2.

Find the ſum of all the poſſible terms of this kind, which call B.

In the ſame manner from the preceding Lemma may be found the aggregates of the contents of every five, ſix, ſeven, &c. roots or values multiplied into each other, which call reſpectively C, D, E, &c; then will the equation required be $x^n * - np(au + bv + ct + ds + \&c.) x^{n-2} - Ax^{n-3} + Bx^{n-4} - Cx^{n-5} + Dx^{n-6} - \&c. = 0.$

From the ſame principles may be deduced the moſt general reduction yet known of equations to others of inferior dimenſions, *e. g.*

Let (X) $x^n + (A + a\sqrt[m]{p} + b\sqrt[m]{p^2} + c\sqrt[m]{p^3} + \dots + s\sqrt[m]{p^{m-2}} + t\sqrt[m]{p^{m-1}}) x^{n-1} + (B + a'\sqrt[m]{p} + b'\sqrt[m]{p^2} + \dots + s'\sqrt[m]{p^{m-2}} + t'\sqrt[m]{p^{m-1}}) x^{n-2} + (C + a''\sqrt[m]{p} + b''\sqrt[m]{p^2} + \&c.) x^{n-3} + \&c. = 0;$ let $\alpha, \beta, \gamma, \delta,$ &c. be the reſpective roots of the equation $z^m - 1 = 0,$ then, from the principles before given, may be formed the different values of the equation X, which being multiplied into each other from the propoſitions beforementioned of the Meditationes Algebraicæ, may be deduced an equation of *nm* dimenſions free from radicals,

whoſe

whose root is x, and which contains mn unknown quantities A, a, b, c, &c. B, a', b', c', &c. c, a'', b'', c'', & p : for one, two or more of these unknown quantities may be assumed any quantities whatever, and thence may be deduced equations of mn dimensions, which may be reduced to equations $x^n + (\text{A} + a\sqrt[m]{p} + b\sqrt[m]{p^2} + c\sqrt[m]{p^3} + \&c.)$ $x^{n-1} + \&c. = 0$ of n dimensions.

In the same manner may be assumed equations, which involve $\sqrt[m]{p}$, $\sqrt[m]{p^2}$, ..., $\sqrt[m]{p^{m-1}}$; $\sqrt[r]{\text{Q}}$, $\sqrt[r]{\text{Q}^2}$, $\sqrt[r]{\text{Q}^3}$, ...$\sqrt[r]{\text{Q}^{r-1}}$; $\sqrt[s]{\tau}$, $\sqrt[s]{\tau^2}$, $\sqrt[s]{\tau^3}$, ..., $\sqrt[s]{\tau^{s-1}}$, &c ; and from so reducing them as to exterminate the irrational quantities, may often be derived equations whose resolutions or reductions are known.

The method of transforming algebraical equations into others, whose roots bear any assignable algebraical (but not exponential) relation to the roots of a given algebraical equation first published by me in the papers sent to the Royal Society, and afterwards in the year 1760 ; and thirdly in my Miscellanea Analytica ; and lastly in the Meditationes Algebraicæ, and since published by Mr. LE GRANGE in the Berlin Acts, is perhaps (as Mr. LE GRANGE observes) more general than Mr. HUDDE's, or any transformation yet invented ; it is very useful in the resolution of numerous problems ; and

further

further has this peculiar advantage over all other trans-
formations yet invented, that it often eafily difcovers
fome of the firft terms of the equation required, from
which many elegant Theorems may be derived.

In the works above-mentioned, viz. Mifcell. Analyt.
Medit. Algeb. &c. are given fome problems ferving to
this transformation; the firft of which is a feries, which
from the coefficients of a given algebraical equation
$(x^n - px^{n-1} + qx^{n-2} - \&c. = 0)$ finds the fum of any power
of the roots (viz. $\alpha^m + \beta^m + \gamma^m + \delta^m + \&c.$ where α, β, γ, δ,
&c. denote the roots of the given equation), the law of
which feries was publifhed by me many years before
that it was given by Mr. EULER.   The third Problem
often mentioned in this paper is an elegant and ufeful
feries for finding the fum of quantities of the follow-
ing kind, viz. $\alpha^m \beta^r \gamma^s \delta^t$, &c. $+ \alpha^r \beta^m \gamma^s \delta^t$, &c. $+ \alpha^t \beta^m \gamma^r \delta^s$ &c.
$+ \alpha^t \beta^m \gamma^r \delta^s$ &c. $+ \alpha^m \beta^t \gamma^r \delta^s$ &c $+$ &c.

Mr. EULER gave the following refolution, $x = \sqrt[n]{\pi} +$
$\sqrt[n]{\rho} + \sqrt[n]{\sigma} + \sqrt[n]{\tau} + \&c.$ where π, ρ, σ, τ, &c. denote the roots
of an equation of $n-1$ dimenfions $v^{n-1} - pv^{n-2} + qv^{n-3} -$
&c. $= 0$.   It is evident, that in this cafe the equation
whofe root is x will have n^{n-1} dimenfions; for let the
roots of the equation $z^n - 1 = 0$ be denoted by α, β, γ, δ,
&c. then will the quantity $\sqrt[n]{\pi}$ have the n following va-
lues

lues $\alpha\sqrt[n]{\pi}$, $\beta\sqrt[n]{\pi}$, $\gamma\sqrt[n]{\pi}$, &c. and the fame may be affirmed of the quantities $\sqrt[n]{\rho}$, $\sqrt[n]{\sigma}$, $\sqrt[n]{\tau}$, &c. and confequently the quantity $\sqrt[n]{\pi}+\sqrt[n]{\rho}$ will have $n\times n$ different values; and in the fame manner the root $x=\sqrt[n]{\pi}+\sqrt[n]{\rho}+\sqrt[n]{\sigma}+\sqrt[n]{\tau}+$ may be proved to contain $n\times n\times n\times n\times$ &c. $=n^{n-1}$ roots, and confequently in this refolution, in equations of fupe-rior dimenfions, the number of independent coefficients $(n-1)$ will be very few in proportion to the number of dimenfions n^{n-1}, or (if we refpect its formula) n^{n-2} of the refulting equation.

Let $n=3$, and the equation refulting will rife to an equation of nine dimenfions, which has the formula of a cubic; for let $x=\sqrt[3]{\pi}+\sqrt[3]{\rho}=a$ one root, then will $\frac{-1+\sqrt{-3}}{2}a$ & $\frac{-1-\sqrt{-3}}{2}a$ be two other of the nine roots, and confequently the roots will be $\overline{x^3-a^3}\times\overline{x^3-b^3}\times\overline{x^3-c^3}=0$, which has the formula of a cubic : and in general the above-mentioned equation of n^{n-1} dimenfions will, for the fame reafon, have the formula of an equation of n^{n-2} dimenfions.

Let the refolution be $x=\sqrt[n]{\pi}+\sqrt[n]{\rho}+\sqrt[n]{\sigma}+\sqrt[n]{\tau}+$ &c. where π, ρ, σ, τ, &c. denote the roots of an equation $x^{n-1}-px^{n-2}+qx^{n-3}-$ &c. $=0$ of $(n-1)$ dimenfions, then will the refulting equation free from radicals, whofe root is x, rife to 2^{n-1} dimenfions; but as every affirma-
tive

tive has a negative root equal to it, it will have the formula of an equation of 2^{n-2} dimensions.

Let the resolution be of this formula $x = \sqrt[r \times m]{\alpha} + \sqrt[r \times m]{\beta} + \sqrt[r \times m]{\gamma} + \sqrt[r \times m]{\delta} +$ &c. if α, β, γ, δ, &c. be considered as the r power of the roots of an equation of s dimensions, then will the resulting equation, of which the resolution is given, rise only to an equation of the formula of m^{s-1} dimensions.

In the year 1762 I published some reasons, for which this method could not extend to the general resolution of algebraical equations.